JOURNAL OF APPROXIMATION THEORY 44, 194–195 (1985)

Letter to the Editor

An Explicit Solution of an L^1 Approximation Problem

D. KERSHAW

Department of Mathematics, University of Lancaster, Lancaster, England

Communicated by E. W. Cheney

Received November 1, 1984

It is a standard result which is proved in [1] that

$$\int_{-1}^{1} |2^{-n}U_n(x)|\, dx < \int_{-1}^{1} |m_n(x)|\, dx,$$

where $2^{-n}U_n$ is the nth degree monic Chebyshev polynomial of the second kind and m_n is any other monic polynomial of degree n. More generally, one could ask for that monic polynomial p_n of degree n for which

$$\int_{-1}^{1} |p_n(x)|\, w(x)\, dx < \int_{-1}^{1} |m_n(x)|\, w(x)\, dx,$$

where w is some nontrivial nonnegative weight function.

Following the theory presented in [1], we see that it is enough to guarantee for p_n the equalities

$$\int_{-1}^{1} x^r \operatorname{sgn}[p_n(x)]\, w(x)\, dx = 0, \qquad r = 0, 1, \ldots, n-1.$$

We shall now show that

$$\int_{-1}^{1} x^r \operatorname{sgn}[T_n(x)](1-x^2)^{-1/2}\, dx = 0, \qquad r = 0, 1, \ldots, n-1, \tag{1}$$

which will give the result that

$$\int_{-1}^{1} |2^{-n+1}T_n(x)|\,(1-x^2)^{-1/2}\, dx < \int_{-1}^{1} |m_n(x)|\,(1-x^2)^{-1/2}\, dx, \tag{2}$$

0021-9045/85 $3.00

where T_n is the nth degree Chebyshev polynomial of the first kind and m_n is any monic polynomial of degree n except $2^{-n+1}T_n$.

Clearly (1) is equivalent to

$$\int_{-1}^{1} T_r(x)\,\mathrm{sgn}[T_n(x)](1-x)^{-1/2}\,dx=0, \qquad r=0, 1,..., n-1,$$

which, with the substitution $x=\cos\theta$, become

$$\int_{0}^{\pi} \cos r\theta.\ \mathrm{sgn}[\cos n\theta]\,d\theta=0, \qquad r=0, 1,..., n-1,$$

namely, the result of problem 7 on page 223 of [1].

Reference

1. W. Cheney, "Introduction to Approximation Theory," Chapter 6, Section 6, McGraw-Hill, New York, 1966.

Printed in Belgium